Int. J. Solids Structures, 1970, Vol. 6, pp. 773 to 784. Pergamon Press. Printed in Great Britain

MATERIAL BODIES WITH UNIFORM SYMMETRY

C.-C. WANG

Department of Mathematical Sciences, Rice University, Houston, Texas

Abstract—A general theory of elastic bodies made up of materials possessing a fixed type of symmetry, such as fluids, isotropic solids, transversely isotropic solids, etc., is formulated in this paper. The specific results include a derivation of the field equations for elastic bodies with uniform symmetry and some statical and dynamical universal solutions for incompressible isotropic solid bodies.

1. INTRODUCTION

IN CONTINUUM mechanics, the classification of materials is based on material symmetry. For example, a material is a *fluid* if its symmetry group relative to any reference configuration is the unimodular group. On the other hand, a material is a *solid* if its symmetry group relative to some reference configuration is a subgroup of the orthogonal group. An *isotropic solid* is a material whose symmetry group relative to some reference configuration coincides with the orthogonal group. In general, we define a *type* of materials as a collection consisting of materials whose symmetry groups relative to some reference configurations coincide with one another. Then fluids form a type, and so do isotropic solids.

In this paper, we formulate a theory of bodies made up of materials belonging to a fixed type, such as fluids, isotropic solids, transversely isotropic solids, etc. We do not require the bodies to be materially uniform, although material uniformity is clearly sufficient for the uniformity of material symmetry. Thus this theory includes as a special case the theory† of materially uniform bodies.

In order to express the field equation in terms of the deformation of the body, we need a *smoothness condition* on the distribution of the constitutive equations of the material particles. We shall set forth this smoothness condition in detail in the next section. Then in Section 3, we derive the field equation for an arbitrary smooth elastic body with uniform symmetry. Finally, some statical and dynamical universal solutions for some incompressible isotropic solid bodies are obtained in Section 4.

The basic concepts of this paper are motivated mainly by the structure of a materially uniform smooth body developed in my papers [2] and [3]. The universal solutions obtained in this paper generalize those of my paper [4].

2. THE CONCEPT OF A SMOOTH MATERIAL BODY WITH UNIFORM SYMMETRY

We use the basic notations of modern continuum mechanics as presented, for example, in Truesdell and Noll [5].

As usual, we assume that the point set of a material body $\mathscr{B}$ forms a *body manifold*, which is a three-dimensional differentiable manifold that can be covered by a global

† Cf. Noll [1] and Wang [2].

coordinate system. Relative to a fixed rectangular Cartesian coordinate system on the physical space, a global coordinate system on $\mathscr{B}$ corresponds to a *configuration* of $\mathscr{B}$, say

$$\boldsymbol{\kappa} : \mathscr{B} \to \mathscr{R}^3. \tag{2.1}$$

Let p be a point in $\mathscr{B}$. We denote the tangent space of $\mathscr{B}$ at p as $\mathscr{B}_p$. Then a *local configuration* of p is a linear isomorphism of $\mathscr{B}_p$ with $\mathscr{R}^3$, say

$$\mathbf{v}_p : \mathscr{B}_p \to \mathscr{R}^3. \tag{2.2}$$

As usual, a configuration $\boldsymbol{\kappa}$ gives rise to a local configuration $\boldsymbol{\kappa}_{*p}$ of p, called the *induced local configuration of* $\boldsymbol{\kappa}$ at p.

In continuum mechanics, the mathematical model for the mechanical response of a material point is the *constitutive equation.* In this paper, we adopt the constitutive equation that defines an *elastic material.* This special model is chosen only for the sake of simplicity. The central concept of this theory, namely material symmetry, is by no means limited to elastic materials.

For an elastic material point p, the constitutive equation is

$$\mathbf{T}(\boldsymbol{\kappa}(p)) = \mathbf{G}(\boldsymbol{\kappa}_{*p}), \tag{2.3}$$

which says that in any configuration $\boldsymbol{\kappa}$ the stress tensor $\mathbf{T}$ at $\boldsymbol{\kappa}(p)$ is a function of the induced local configuration of $\boldsymbol{\kappa}$ at p. We call $\mathbf{G}$ the *response function* of p. From (2.3), we see that a local configuration of p corresponds to a *local experiment* at p, the outcome of which being the stress tensor. This interpretation leads us to a definition of the *material symmetry* of p in a natural way:

Suppose that $\mathbf{A}$ is an automorphism of the tangent space $\mathscr{B}_p$,

$$\mathbf{A} : \mathscr{B}_p \to \mathscr{B}_p. \tag{2.4}$$

Then we say that $\mathbf{A}$ is a *material automorphism* of p if the outcome of any local experiment $\mathbf{v}_p$ is the same as that of $\mathbf{v}_p \circ \mathbf{A}$, i.e.

$$\mathbf{G}(\mathbf{v}_p) \equiv \mathbf{G}(\mathbf{v}_p \circ \mathbf{A}), \qquad \forall \mathbf{v}_p. \tag{2.5}$$

In other words, a material automorphism of p is a transformation that cannot be detected by any local experiment at p.

The collection of all material automorphisms of p forms a group $\mathscr{g}_p$, called the *abstract symmetry group* of p. In general, a material automorphism $\mathbf{A}$ must be volume-preserving, since otherwise the stress can remain unchanged when the body is compressed or dilated by any power $\mathbf{A}^n$ or $\mathbf{A}^{-n}$, but such behavior corresponds to no known physical body. Hence $\mathscr{g}_p$ is a subgroup of the unimodular group of $\mathscr{B}_p$.

We define a *reference configuration* and a *local reference configuration* as any distinguished configuration and any distinguished local configuration, respectively. Relative to a fixed local reference configuration $\boldsymbol{\mu}_p$, we can factorize any local configuration $\mathbf{v}_p$ in the following way:

$$\begin{array}{ccc} \mathscr{B}_p & \xrightarrow{\ \boldsymbol{\mu}_p\ } & \mathscr{R}^3 \\ & \searrow^{\mathbf{v}_p} & \downarrow \mathbf{F} \\ & & \mathscr{R}^3 \end{array} \tag{2.6}$$

where $\mathbf{F}$ is given by

$$\mathbf{F} = \mathbf{v}_p \circ \boldsymbol{\mu}_p^{-1}, \tag{2.7}$$

called the *deformation gradient* from $\boldsymbol{\mu}_p$ to $\mathbf{v}_p$. Using the deformation gradient $\mathbf{F}$, we can rewrite the constitutive equation (2.3) as

$$\mathbf{T} = \mathbf{H}(\mathbf{F}, p), \tag{2.8}$$

and we call $\mathbf{H}$ the *response function of p relative to* $\boldsymbol{\mu}_p$.

Now let $\bar{\boldsymbol{\mu}}_p$ be another local reference configuration, and let $\bar{\mathbf{H}}$ be its corresponding relative response function. Then $\bar{\mathbf{H}}$ is related to $\mathbf{H}$ by

$$\bar{\mathbf{H}}(\mathbf{F}, p) = \mathbf{H}(\mathbf{F}\mathbf{P}, p), \tag{2.9}$$

where $\mathbf{P}$ is the deformation gradient from $\boldsymbol{\mu}_p$ to $\bar{\boldsymbol{\mu}}_p$, viz

$$\mathbf{P} = \bar{\boldsymbol{\mu}}_p \circ \boldsymbol{\mu}_p^{-1}. \tag{2.10}$$

We say that $\bar{\boldsymbol{\mu}}_p$ is *materially isomorphic* to $\boldsymbol{\mu}_p$ if $\bar{\mathbf{H}}$ coincides with $\mathbf{H}$. From (2.9), we see that $\bar{\boldsymbol{\mu}}_p$ is materially isomorphic to $\boldsymbol{\mu}_p$ if and only if

$$\mathbf{H}(\mathbf{F}, p) = \mathbf{H}(\mathbf{F}\mathbf{P}, p), \qquad \forall \mathbf{F}. \tag{2.11}$$

A tensor $\mathbf{P}$ satisfying this condition is called a *material automorphism of p relative to* $\boldsymbol{\mu}_p$. The collection of all such material automorphisms form a group $\mathscr{g}_p(\boldsymbol{\mu}_p)$, called the *symmetry group of p relative to* $\boldsymbol{\mu}_p$.

The meaning of the condition (2.11) is similar to that of (2.5). Relative to the fixed local reference configuration $\boldsymbol{\mu}_p$ a local experiment of p can be characterized by the deformation gradient $\mathbf{F}$. Then (2.11) says that a transformation $\mathbf{P}$ is a material automorphism relative to $\boldsymbol{\mu}_p$ if it cannot be detected by any local experiment relative to $\boldsymbol{\mu}_p$.

From (2.6), we have

$$\mathbf{H}(\mathbf{F}, p) = \mathbf{G}(\mathbf{F} \circ \boldsymbol{\mu}_p), \tag{2.12}$$

so that $\mathbf{P}$ belongs to $\mathscr{g}_p(\boldsymbol{\mu}_p)$ if and only if $\boldsymbol{\mu}_p^{-1} \circ \mathbf{P} \circ \boldsymbol{\mu}_p$ belongs to $\mathscr{g}_p$. Thus in the set-theoretical sense

$$\mathscr{g}_p(\boldsymbol{\mu}_p) = \boldsymbol{\mu}_p \circ \mathscr{g}_p \circ \boldsymbol{\mu}_p^{-1}. \tag{2.13}$$

This equation implies directly the transformation rule:

$$\mathscr{g}_p(\bar{\boldsymbol{\mu}}_p) = \mathbf{P}\mathscr{g}_p(\boldsymbol{\mu}_p)\mathbf{P}^{-1}, \tag{2.14}$$

where $\mathbf{P}$ is the deformation gradient from $\boldsymbol{\mu}_p$ to $\bar{\boldsymbol{\mu}}_p$. From (2.13), we see that $\mathscr{g}_p(\boldsymbol{\mu}_p)$ is a subgroup of the unimodular group of $\mathscr{R}^3$. Then from (2.14), the relative symmetry groups of p are inner isomorphic to one another within the unimodular group. In group theory, an equivalence class of inner isomorphic subgroups is called a *type*. Thus the relative symmetry groups of a material point form a type of subgroups of the unimodular group.

We say that $\mathscr{B}$ is a *material body with uniform symmetry* if the types of relative symmetry groups of the points of $\mathscr{B}$ coincide with one another. Equivalently, this condition means that for any two points p and q in $\mathscr{B}$, there exists a linear isomorphism

$$\mathbf{r}(p, q): \mathscr{B}_p \to \mathscr{B}_q, \tag{2.15}$$

called a *symmetry isomorphism of p with q*, such that

$$\mathscr{g}_q = \mathbf{r}(p,q) \circ \mathscr{g}_p \circ \mathbf{r}(p,q)^{-1}. \tag{2.16}$$

Indeed, if $\boldsymbol{\mu}_p$ and $\boldsymbol{\mu}_q$ are local reference configurations of p and q respectively, such that

$$\mathscr{g}_p(\boldsymbol{\mu}_p) = \mathscr{g}_q(\boldsymbol{\mu}_q), \tag{2.17}$$

then the composite map

$$\mathbf{r}(p,q) = \boldsymbol{\mu}_q^{-1} \circ \boldsymbol{\mu}_p \tag{2.18}$$

satisfies the condition (2.16). Conversely, if $\boldsymbol{\mu}_p$ and $\boldsymbol{\mu}_q$ are related by (2.18), with $\mathbf{r}(p,q)$ satisfying the condition (2.16), then (2.17) holds. A sufficient but not necessary condition for a symmetry isomorphism $\mathbf{r}(p,q)$ is

$$\mathbf{G}[\mathbf{v}_q \circ \mathbf{r}(p,q)] = \mathbf{G}(\mathbf{v}_q), \qquad \forall \mathbf{v}_q. \tag{2.19}$$

If this condition is satisfied, p and q are said to be *materially isomorphic*, and $\mathbf{r}(p,q)$ is called a *material isomorphism of p with q*. Substituting (2.19) into (2.5) yields

$$\mathbf{G}(\mathbf{v}_q) = \mathbf{G}[\mathbf{v}_q \circ \mathbf{r}(p,q) \circ \mathbf{A} \circ \mathbf{r}(p,q)^{-1}], \qquad \forall \mathbf{v}_q, \tag{2.20}$$

which means that $\mathbf{r}(p,q)$ satisfies (2.16). Thus a material isomorphism is a symmetry isomorphism, but the converse is not true in general.

In short, a material body with uniform symmetry is a body whose points are mutually symmetry-isomorphic.

Next, we set forth a smoothness condition on a body with uniform symmetry. We define a *symmetry chart* as a pair $(\mathscr{U}, \boldsymbol{\mu})$ consisting of a subbody $\mathscr{U}$ of $\mathscr{B}$ and a smooth field $\boldsymbol{\mu}$ of local configurations on $\mathscr{U}$, such that the symmetry groups relative to $\boldsymbol{\mu}$ are independent of the points in $\mathscr{U}$. We put

$$\mathscr{g}(\mathscr{U}, \boldsymbol{\mu}) = \mathscr{g}_p(\boldsymbol{\mu}_p), \qquad \forall p \in \mathscr{U}, \tag{2.21}$$

called the *symmetry group of $\mathscr{U}$ relative to $\boldsymbol{\mu}$*. *Note.* Since $\boldsymbol{\mu}$ is required to be smooth, in general it need not have any global extension on the whole body $\mathscr{B}$. Thus the existence of a symmetry chart is a local condition.

We say that two symmetry charts $(\mathscr{U}, \boldsymbol{\mu})$ and $(\overline{\mathscr{U}}, \bar{\boldsymbol{\mu}})$ are *compatible* if the overlap $\mathscr{U} \cap \overline{\mathscr{U}}$ is non-empty, and if the response functions relative to $\boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ coincide on $\mathscr{U} \cap \overline{\mathscr{U}}$. Equivalently, this condition means

$$\boldsymbol{\mu}_p^{-1} \circ \bar{\boldsymbol{\mu}}_p \in \mathscr{g}_p, \qquad \forall p \in \mathscr{U} \cap \overline{\mathscr{U}}. \tag{2.22}$$

Of course, if $(\mathscr{U}, \boldsymbol{\mu})$ is compatible with $(\overline{\mathscr{U}}, \bar{\boldsymbol{\mu}})$, then

$$\mathscr{g}(\mathscr{U}, \boldsymbol{\mu}) = \mathscr{g}(\overline{\mathscr{U}}, \bar{\boldsymbol{\mu}}), \tag{2.23}$$

but the converse is not true in general. We define the field

$$\mathbf{G}(p) = \boldsymbol{\mu}_p \circ \bar{\boldsymbol{\mu}}_p^{-1}, \qquad p \in \mathscr{U} \cap \overline{\mathscr{U}}, \tag{2.24}$$

as the *coordinate transformation* from $(\mathscr{U}, \boldsymbol{\mu})$ to $(\overline{\mathscr{U}}, \bar{\boldsymbol{\mu}})$. From (2.22), $\mathbf{G}$ has values in the relative symmetry group $\mathscr{g}(\mathscr{U}, \boldsymbol{\mu})$. Suppose that the coordinate transformation $\mathbf{G}$ is the identity transformation on $\mathscr{U} \cap \overline{\mathscr{U}}$. Then $\boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ agree on the overlap of their domains. In this case, we can regard $\bar{\boldsymbol{\mu}}$ as a smooth extension of $\boldsymbol{\mu}$ to the bigger subbody $\mathscr{U} \cup \overline{\mathscr{U}}$. On the other hand, if $\mathbf{G}$ is not the identity transformation, then $\boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ are different fields on $\mathscr{U} \cap \bar{\mathscr{U}}$. But in any case, the relative response functions on $\mathscr{U} \cap \bar{\mathscr{U}}$ associated with $\boldsymbol{\mu}$ are the same as those associated with $\bar{\boldsymbol{\mu}}$. Unlike the condition of a symmetry chart, which characterizes the smoothness of the local distribution of the relative response functions, the condition of compatibility requires that the distribution of the relative response functions on $\mathscr{U}$ can be extended to a distribution on $\mathscr{U} \cup \bar{\mathscr{U}}$. In this sense, the existence of a covering of $\mathscr{B}$ by mutually compatible symmetry charts characterizes a smooth global distribution of relative response functions on $\mathscr{B}$.

More specifically, we define a *symmetry atlas* $\mathfrak{A}$ of $\mathscr{B}$ as a maximal collection of mutually compatible symmetry charts covering $\mathscr{B}$, say

$$\mathfrak{A} = \{(\mathscr{U}_\alpha, \boldsymbol{\mu}_\alpha), \alpha \in I\}, \tag{2.25}$$

where I is an index set. Then the following three conditions are satisfied:

(i) There exists a smooth distribution of relative response functions $\mathbf{H}_{\mathfrak{A}}(\mathbf{F}, p)$ on $\mathscr{B}$ with

$$\mathbf{H}_{\mathfrak{A}}(\mathbf{F}, p) = \mathbf{G}(\mathbf{F} \circ \boldsymbol{\mu}_{\alpha p}), \qquad \forall \mathbf{F} \tag{2.26}$$

for all symmetry charts $(\mathscr{U}_\alpha, \boldsymbol{\mu}_\alpha)$ in $\mathfrak{A}$ covering p.

(ii) The relative response functions $\mathbf{H}_{\mathfrak{A}}(\mathbf{F}, p)$ have the uniform relative symmetry group, namely

$$\mathscr{g}(\mathfrak{A}) = \mathscr{g}(\mathscr{U}_\alpha, \boldsymbol{\mu}_\alpha), \qquad \forall \alpha \in I. \tag{2.27}$$

(iii) The coordinate transformation

$$\mathbf{G}_{\alpha\beta} = \boldsymbol{\mu}_\alpha \circ \boldsymbol{\mu}_\beta^{-1} \tag{2.28}$$

from $(\mathscr{U}_\alpha, \boldsymbol{\mu}_\alpha)$ to $(\mathscr{U}_\beta, \boldsymbol{\mu}_\beta)$ is a smooth field with values in $\mathscr{g}(\mathfrak{A})$ on the overlap $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$, for all $\alpha, \beta, \in I$.

Naturally, we say that $\mathscr{B}$ is *smooth* if it can be equipped with a symmetry atlas $\mathfrak{A}$. In application, we often assume that the response function $\mathbf{G}$ for each particle is smooth also. Under this additional assumption, the distribution $\mathbf{H}_{\mathfrak{A}}$ is smooth jointly in $\mathbf{F}$ and p, and its symmetry group $\mathscr{g}(\mathfrak{A})$ is a closed Lie subgroup of the unimodular group over $\mathscr{R}^3$.

As we explained before, a smooth materially uniform body is a body with uniform symmetry. Indeed, a material atlas† of a materially uniform body is a symmetry atlas, but the converse is not true in general. A necessary and sufficient condition for a symmetry atlas $\mathfrak{A}$ to be a material atlas is that the distribution $\mathbf{H}_{\mathfrak{A}}$ be independent of the material points of $\mathscr{B}$.

For a smooth body $\mathscr{B}$ with uniform symmetry, we call $\mathbf{H}_{\mathfrak{A}}$ the *distribution of response functions relative to* $\mathfrak{A}$, and $\mathscr{g}(\mathfrak{A})$ the *symmetry group relative to* $\mathfrak{A}$. The symmetry atlas $\mathfrak{A}$ of $\mathscr{B}$ is not unique, of course. From the maximality condition on a symmetry atlas, we have the criterion

$$\mathfrak{A} = \bar{\mathfrak{A}} \Leftrightarrow \mathbf{H}_{\mathfrak{A}} = \mathbf{H}_{\bar{\mathfrak{A}}}. \tag{2.29}$$

Thus each symmetry atlas is characterized by its corresponding distribution of the relative response functions. The general linear group of $\mathscr{R}^3$ is a transformation group on the symmetry atlases of $\mathscr{B}$. Let $\mathbf{P}$ be a non-singular transformation of $\mathscr{R}^3$. We put

$$\mathbf{P}\mathfrak{A} = \{(\mathscr{U}_\alpha, \mathbf{P} \circ \boldsymbol{\mu}_\alpha), \alpha \in I\}. \tag{2.30}$$

† Cf. Wang [2].

Then $\mathbf{P}\mathfrak{A}$ is a symmetry atlas; further, the transformation rules for $\mathbf{H}_{\mathfrak{A}}$ and $\mathscr{g}(\mathfrak{A})$ under such an operation $\mathbf{P}$ are

$$\mathbf{H}_{\mathbf{P}\mathfrak{A}}(\mathbf{F}, p) = \mathbf{H}_{\mathfrak{A}}(\mathbf{FP}, p), \qquad \forall \mathbf{F}, p, \tag{2.31}$$

and

$$\mathscr{g}(\mathbf{P}\mathfrak{A}) = \mathbf{P}\mathscr{g}(\mathfrak{A})\mathbf{P}^{-1}. \tag{2.32}$$

In particular, we can characterize the relative symmetry group $\mathscr{g}(\mathfrak{A})$ by the condition

$$\mathbf{P} \in \mathbf{g}_{\mathfrak{A}} \Leftrightarrow \mathbf{H}_{\mathbf{P}\mathfrak{A}} = \mathbf{H}_{\mathfrak{A}} \Leftrightarrow \mathbf{P}\mathfrak{A} = \mathfrak{A}. \tag{2.33}$$

Notice, however, the operation $\mathbf{P}$ is not transitive on the symmetry atlases. In general, two symmetry atlases $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ need not be related by any such operation $\mathbf{P}$.

The geometric structure on $\mathscr{B}$ determined by a symmetry atlas $\mathfrak{A}$ is similar to that determined by a material atlas. A detailed analysis of that structure has been worked out in Ref. [2]. In that structure, an important element is a structural connection, called a *material connection* in [2], with respect to $\mathscr{g}(\mathfrak{A})$ as the structure group. For a smooth body $\mathscr{B}$ with uniform symmetry, such a structural connection relative to $\mathscr{g}(\mathfrak{A})$ may be called a *symmetry connection*, since its parallel transports are all symmetry isomorphisms† among the particles of $\mathscr{B}$.

In Ref. [2], we have derived the following necessary and sufficient condition for a structural connection with respect to the structure group $\mathscr{g}(\mathfrak{A})$: Let (x^1, x^2, x^3) be a coordinate system on $\mathscr{B}$, and let Γ^i_{jk} be the connection symbols of a connection $\mathscr{H}$ with respect to (x^i). Then $\mathscr{H}$ is a structural connection with respect to the structure group $\mathscr{g}(\mathfrak{A})$ if, and only if, the matrices

$$\left\{ \left[F^{-1}{}^i_j \left(\frac{\partial F^j_k}{\partial x^m} + \Gamma^j_{lm} F^l_k \right) \right], m = 1, 2, 3 \right\} \tag{2.34}$$

are contained in the Lie algebra of $\mathscr{g}(\mathfrak{A})$. Here F^i_j denotes the components of the deformation gradient $\mathbf{F}$ from $\boldsymbol{\mu}$ to (x^i) for any symmetry chart‡ $(\mathscr{U}, \boldsymbol{\mu})$ in $\mathfrak{A}$.

As we shall see in the next section, the condition (2.34) enables us to express the field equations of $\mathscr{B}$ in global form.

3. THE FIELD EQUATION FOR A MATERIAL BODY WITH UNIFORM SYMMETRY

In this section we derive the equation of motion for a smooth elastic body $\mathscr{B}$ with uniform symmetry. We choose a fixed symmetry atlas $\mathfrak{A}$ for $\mathscr{B}$. For simplicity, we suppress the dependence on $\mathfrak{A}$ from the notations of the relative response functions and their symmetry group. We put

$$H^{ij}{}_{kl} = H^{ij}{}_{kl}(\mathbf{F}, p) \equiv \frac{\partial H^{ij}}{\partial F^{kl}}, \tag{3.1}$$

where the components are taken relative to a fixed rectangular Cartesian coordinate system on the physical space. The order of the indices is important here, since it is understood that the indices may be raised or lowered with respect to the physical Euclidean

† This condition alone is not sufficient for a symmetry connection relative to $\mathscr{g}(\mathfrak{A})$, however.

‡ We proved in [2] that the condition (2.34) is independent of the choice of the symmetry chart in $\mathfrak{A}$.

metric. From the condition (ii) of the symmetry atlas, $\mathbf{H}$ satisfies the symmetry condition

$$\mathbf{H}(\mathbf{FP}, p) = \mathbf{H}(\mathbf{F}, p), \qquad \forall \mathbf{F}, p, \tag{3.2}$$

for all $\mathbf{P} \in \mathscr{g}$. Then the gradient of $\mathbf{H}$ satisfies the condition

$$H^{ij}{}_{kl}(\mathbf{F}, p)F^{km}K^{l}_{m} = 0 \tag{3.3}$$

for all $\mathbf{K}$ belonging to the Lie algebra of $\mathscr{g}$, and

$$H^{ij}{}_{kl}(\mathbf{F}, p) = H^{ij}{}_{km}(\mathbf{FP}, p)P^{m}_{l} \tag{3.4}$$

for all $\mathbf{P}$ belonging to $\mathscr{g}$. These two conditions are obtained by differentiating (3.2) with respect to $\mathbf{P}$ and $\mathbf{F}$, respectively.

Now let χ be a configuration of $\mathscr{B}$ corresponding to the coordinate system (x^1, x^2, x^3). Then the stress tensor in $\chi(\mathscr{B})$ can be determined in the following way: First, we choose a symmetry chart $(\mathscr{U}_\alpha, \boldsymbol{\mu}_\alpha)$ in $\mathfrak{A}$. Then for any point p covered by $\boldsymbol{\mu}_\alpha$, the deformation gradient from $\boldsymbol{\mu}_{\alpha p}$ to χ_{*p} is

$$\mathbf{F} = \chi_{*p} \circ \boldsymbol{\mu}_{\alpha p}^{-1}. \tag{3.5}$$

Substituting (3.5) into the distribution of the relative response functions $\mathbf{H}$ yields the stress tensor

$$\mathbf{T}[\chi(p)] = \mathbf{H}(\chi_{*p} \circ \boldsymbol{\mu}_{\alpha p}^{-1}, p). \tag{3.6}$$

Of course, this is a local expression, since the field $\boldsymbol{\mu}_\alpha$ is defined on the subbody $\mathscr{U}_\alpha$ only.

From the principle of linear momentum, the governing equation of motion for a continuum is

$$\operatorname{Div} \mathbf{T} + \rho \mathbf{b} = \rho \mathbf{a}, \tag{3.7}$$

where ρ is the density, $\mathbf{b}$ is the body force (including the inertia force if the frame of reference is not an inertial one), and $\mathbf{a}$ is the acceleration field. Substituting (3.6) into (3.7), we obtain

$$H^{ij}{}_{kl}\frac{\partial F^{kl}}{\partial x^j} + H^{ij}{}_{j} + \rho b^i = \rho a^i, \tag{3.8}$$

where F^{kl} denotes the components of $\mathbf{F}$ in (x^i), and where

$$H^{ij}{}_{k} = H^{ij}{}_{k}(\mathbf{F}, x^m(p)) = \frac{\partial H^{ij}}{\partial x^k}. \tag{3.9}$$

The arguments of $H^{ij}{}_{k}$ and $H^{ij}{}_{j}$ in (3.8), of course, are $\mathbf{F}$ and p, the latter being characterized by the coordinates (x^1, x^2, x^3). Like $H^{ij}{}_{kl}$, the function $H^{ij}{}_{k}$ obeys the symmetry condition

$$H^{ij}{}_{k}(\mathbf{FP}, x^m) = H^{ij}{}_{k}(\mathbf{F}, x^m), \qquad \forall \mathbf{F}, \tag{3.10}$$

for all $\mathbf{P}$ belonging to $\mathscr{g}$. From this condition and the condition (iii) of the symmetry atlas $\mathfrak{A}$, we see that the field $H^{ij}{}_{k}$ is independent of the choice of the symmetry chart. Thus in the field equation (3.8), only the leading term is a local field. Now using a technique developed in Ref. [2], we can replace that leading term by a global field also, as follows:

We choose a symmetry connection $\mathscr{H}$ associated with the symmetry atlas $\mathfrak{A}$. Then the connection symbols Γ^i_{jk} of $\mathscr{H}$ in (x^i) satisfy the condition (2.34). Substituting that condition into (3.3), we obtain

$$H^{ij}{}_{kl}\left(\frac{\partial F^{kl}}{\partial x^m}+\Gamma^k_{nm}F^{nl}\right) = 0. \tag{3.11}$$

Hence the field equation takes also the form

$$-H^{ij}{}_{kl}F^{nl}\Gamma^k_{nj}+H^{ij}{}_{j}+\rho b^i = \rho a^i. \tag{3.12}$$

Now the leading term is a global field, since from (3.4) and (2.27), the field

$$\bar{H}^{ij}{}_{k}{}^{n} \equiv H^{ij}{}_{kl}(\mathbf{F}, p)F^{nl}, \tag{3.13}$$

like the field $H^{ij}{}_{k}$, is independent of the choice of the symmetry chart.

In application, the field equation (3.12) is not convenient, since the connection symbols Γ^i_{jk} depend on the coordinate system (x^1, x^2, x^3) of χ, which changes with time. To render the time dependence of Γ^i_{jk} explicit, we introduce a fixed reference configuration $\boldsymbol{\kappa}$ with coordinates (X^1, X^2, X^3) on $\mathscr{B}$. Then a motion of $\mathscr{B}$ can be expressed by the deformation functions

$$x^i = x^i(X^A, t), \tag{3.14}$$

which merely correspond to a 1-parameter family of change of coordinates. Now let the connection symbols of $\mathscr{H}$ relative to (X^A) be $\bar{\Gamma}^A_{BC}$, and let the components of the deformation gradient from $\boldsymbol{\mu}$ to $\boldsymbol{\kappa}$ be $\bar{F}^{Ai}$. Then we have the usual transformation rules:

$$\Gamma^i_{jk} = \bar{\Gamma}^A_{BC}\frac{\partial x^i}{\partial X^A}\frac{\partial X^B}{\partial x^j}\frac{\partial X^C}{\partial x^k}-\frac{\partial^2 x^i}{\partial X^A\partial X^B}\frac{\partial X^A}{\partial x^j}\frac{\partial X^B}{\partial x^k}, \tag{3.15}$$

and

$$F^{ij} = \frac{\partial x^i}{\partial X^A}\bar{F}^{Aj}. \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.12), we obtain

$$\tilde{H}^{ij}{}_{k}{}^{A}\frac{\partial X^B}{\partial x^j}\left(\frac{\partial^2 x^k}{\partial X^A\partial X^B}-\bar{\Gamma}^C_{AB}\frac{\partial x^k}{\partial X^C}\right)+H^{ij}{}_{j}+\rho b^i = \rho a^i, \tag{3.17}$$

where locally

$$\tilde{H}^{ij}{}_{k}{}^{A} = H^{ij}{}_{kl}(\mathbf{F}, p)\bar{F}^{Al}. \tag{3.18}$$

The equation (3.17) is a global field equation of motion for the body $\mathscr{B}$.

We can express the field equation in terms of the Piola–Kirchhoff stress tensor† $\mathbf{T}_{\boldsymbol{\kappa}}$ relative to a fixed reference configuration $\boldsymbol{\kappa}$ also. By definition, $\mathbf{T}$ and $\mathbf{T}_{\boldsymbol{\kappa}}$ are related by

$$T^{kA}_{\boldsymbol{\kappa}} = JT^{kl}\frac{\partial X^A}{\partial x^l}, \tag{3.19}$$

where J denotes the determinant of the deformation gradient,

$$J = \det\left[\frac{\partial x^i}{\partial X^A}\right]. \tag{3.20}$$

† Also known as the *engineering stress tensor*, cf. Truesdell and Noll [5, Section 43A].

In terms of $\mathbf{T}_\kappa$, the principle of linear momentum takes the form

$$\operatorname{Div} \mathbf{T}_\kappa + \rho_\kappa \mathbf{b} = \rho_\kappa \mathbf{a}, \tag{3.21}$$

where Div denotes the divergence with respect to $\boldsymbol{\kappa}$, and where ρ_κ is the density field of $\boldsymbol{\kappa}(\mathscr{B})$.

We define the Piola–Kirchhoff response function $\mathbf{A}$ relative to $\mathfrak{A}$ by

$$\mathbf{A} = \mathbf{A}(\mathbf{F}, p) \equiv (\det \mathbf{F})\mathbf{H}(\mathbf{F}, p)(\mathbf{F}^{-1})^{\mathbf{T}} \tag{3.22}$$

where the superscript $\mathbf{T}$ denotes the transposition. In component form, (3.22) takes the form

$$A^{ik} = (\det \mathbf{F}) H^{ij} F^{-1k}_{\ \ j}, \tag{3.23}$$

or equivalently

$$H^{ij} = \frac{1}{\det \mathbf{F}} A^{ik} F^j_k. \tag{3.24}$$

Differentiating this relation with respect to $\mathbf{F}$ yields

$$H^{ij}{}_{kl} = \frac{1}{\det \mathbf{F}}[F^j_m A^{im}{}_{kl} + (\delta^j_k \delta_{lm} - F^j_m F^{-1}_{lk}) A^{im}], \tag{3.25}$$

where

$$A^{ij}{}_{kl} = \frac{\partial A^{ij}}{\partial F^{kl}}. \tag{3.26}$$

Now substituting (3.25) into (3.17), we obtain

$$\tilde{A}^{iA}{}_k{}^B \left(\frac{\partial^2 x^k}{\partial X^A \partial X^B} - \bar{\Gamma}^C_{BA} \frac{\partial x^k}{\partial X^C} \right) - T^{iA}_\kappa \bar{C}^B_{AB} + J H^{ij}{}_{,j} + \rho_\kappa b^i = \rho_\kappa a^i, \tag{3.27}$$

where locally

$$\tilde{A}^{iA}{}_k{}^B = \frac{1}{\det \bar{\mathbf{F}}} A^{is}{}_{kl}(\mathbf{F}, p) \bar{F}^A_s \bar{F}^{Bl}, \tag{3.28}$$

$$\bar{C}^A_{BC} = \bar{\Gamma}^A_{BC} - \bar{\Gamma}^A_{CB}, \tag{3.29}$$

and

$$T^{iA}_\kappa = \frac{1}{\det \bar{\mathbf{F}}} A^{is}(\mathbf{F}, p) \bar{F}^A_s = J H^{ij}(\mathbf{F}, p) \frac{\partial X^A}{\partial x^j}. \tag{3.30}$$

The equation (3.27) is another global field equation of motion for the body $\mathscr{B}$.

Note. In (3.29), the quantities $\bar{C}^A_{BC}$ are the components of the torsion tensor $\mathbf{C}$ of the connection $\mathscr{H}$ relative to the referential coordinate system (X^A). In general, a symmetry connection $\mathscr{H}$ has non-vanishing torsion and curvature. But unlike the torsion tensor of $\mathscr{H}$, the curvature tensor does not enter explicitly into the field equation. Notice also, the leading term in parentheses of the field equation (3.27) is the covariant derivative of the deformation gradient from $\boldsymbol{\kappa}$ to $\boldsymbol{\chi}$ relative to the connection $\mathscr{H}$.

In the next section, we shall work out some universal solutions for some specific elastic bodies with uniform symmetry.

4. STATICAL AND DYNAMICAL UNIVERSAL SOLUTIONS FOR SOME ISOTROPIC SOLID BODIES

In this section we consider isotropic solid bodies. For such a body $\mathscr{B}$, there exists a symmetry atlas $\mathfrak{A}$ with $\mathscr{g}(\mathfrak{A})$ equal to the orthogonal group $\mathscr{o}$ of $\mathscr{R}^3$. For definiteness, we say that such a symmetry atlas $\mathfrak{A}$ is *undistorted*.

In Section 2, we proved that in general a symmetry atlas $\mathfrak{A}$ is characterized by its corresponding distribution of relative response functions $\mathbf{H}_{\mathfrak{A}}$. For a undistorted symmetry atlas $\mathfrak{A}$ of an isotropic body $\mathscr{B}$, we claim that $\mathfrak{A}$ can be characterized also by a Riemannian metric $\mathbf{g}_{\mathfrak{A}}$. Specifically, we define $\mathbf{g}_{\mathfrak{A}}$ in the following way: For any point $p \in \mathscr{B}$, let $(\mathscr{U}_\alpha, \boldsymbol{\mu}_\alpha)$ by a symmetry chart in $\mathfrak{A}$ covering p. Then we put

$$\mathbf{g}_{\mathfrak{A}p}(\mathbf{u}, \mathbf{v}) \equiv \boldsymbol{\mu}_{\alpha p}(\mathbf{u}) \cdot \boldsymbol{\mu}_{\alpha p}(\mathbf{v}), \tag{4.1}$$

for all $\mathbf{u}, \mathbf{v} \in \mathscr{B}_p$. Here the dot product on the right-hand side is that of the Euclidean metric of the physical space. Of course, we must verify that the inner product $\mathbf{g}_{\mathfrak{A}}$ defined by (4.1) is independent of the choice of the symmetry chart. But this fact is obvious, since from (2.27), with $\mathscr{g}(\mathfrak{A}) = \mathscr{o}$, we have

$$\begin{aligned}\boldsymbol{\mu}_{\beta p}(\mathbf{u}) \cdot \boldsymbol{\mu}_{\beta p}(\mathbf{v}) &= \mathbf{G}_{\beta\alpha}(p)\boldsymbol{\mu}_{\alpha p}(\mathbf{u}) \cdot \mathbf{G}_{\beta\alpha}(p)\boldsymbol{\mu}_{\alpha p}(\mathbf{v}) \\ &= \boldsymbol{\mu}_{\alpha p}(\mathbf{u}) \cdot \boldsymbol{\mu}_{\alpha p}(\mathbf{v}).\end{aligned} \tag{4.2}$$

Thus $\mathbf{g}_{\mathfrak{A}}$ is well-defined. Further, by the maximality condition of a symmetry atlas, two undistorted symmetry atlases $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ coincide if and only if so do their corresponding Riemannian metrics. Indeed, if $\mathbf{g}_{\mathfrak{A}}$ is given, then $\mathfrak{A}$ is the collection consisting of all symmetry charts which satisfy the condition (4.1). Thus $\mathfrak{A}$ is characterized by $\mathbf{g}_{\mathfrak{A}}$.

In Ref. [4], we considered a class of materially uniform isotropic bodies, called *laminated bodies*, whose intrinsic Riemannian metric $\mathbf{g}_{\mathfrak{A}}$ satisfies the following condition: $\mathscr{B}$ can be decomposed into a disjoint union of two-dimensional submanifolds $\mathscr{L}_\xi$, called the *laminae* of $\mathscr{B}$, such that for each $\mathscr{L}_\xi$, locally, there exists a configuration $\boldsymbol{\kappa}_\xi$, whose induced local configuration at $\mathscr{L}_\xi$ carries the metric $\mathbf{g}_{\mathfrak{A}}$ onto the Euclidean metric of the physical space, viz,

$$\mathbf{g}_{\mathfrak{A}p}(\mathbf{u}, \mathbf{v}) = \boldsymbol{\kappa}_{\xi p}(\mathbf{u}) \cdot \boldsymbol{\kappa}_{\xi p}(\mathbf{v}) \tag{4.3}$$

for all p belonging to $\mathscr{L}_\xi \cap \mathscr{U}$, where $\mathscr{U}$ is the domain of $\boldsymbol{\kappa}_\xi$. In the physical interpretation, we can regard a lamina $\mathscr{L}_\xi$ as a locally homogeneous, infinitely thin shell, with $\boldsymbol{\kappa}_\xi$ as its local homogeneous configuration. In Ref. [4], the laminae were required to be materially isomorphic to one another. We now choose a weaker condition that each lamina of $\mathscr{B}$ be materially uniform, but different laminae be only symmetry-isomorphic rather than materially isomorphic.

Specifically, we consider the following three classes of laminated bodies:

(i) *Laminated plate*

This body is characterized by the metric $\mathbf{g}_{\mathfrak{A}}$ whose physical components in a referential rectangular Cartesian coordinates (X, Y, Z) form the matrix

$$[\bar{g}\langle AB\rangle] = \begin{bmatrix} \bar{g}\langle 11\rangle, & 0, & 0 \\ 0, & \bar{g}\langle 22\rangle, & \bar{g}\langle 23\rangle \\ 0, & \bar{g}\langle 32\rangle, & \bar{g}\langle 33\rangle \end{bmatrix}, \tag{4.4}$$

where the non-zero components $\bar{g}\langle AB\rangle$ are functions of X only. Clearly, the laminae are the planes with constant X.

(ii) *Laminated cylindrical shell*

Relative to a cylindrical coordinate system (R, Θ, Z) in the reference configuration $\boldsymbol{\kappa}$, the metric $\mathbf{g}_{\mathfrak{A}}$ has the component form (4.4) with $\bar{g}\langle AB\rangle$ depending only on R. In this case, the laminae are the cylinders with constant R.

(iii) *Laminated spherical shell*

Relative to a spherical coordinate system (R, Θ, Φ) in $\boldsymbol{\kappa}$, the metric $\mathbf{g}_{\mathfrak{A}}$ has the physical component form

$$[\bar{g}\langle AB\rangle] = \mathrm{diag}[\bar{g}\langle 11\rangle, \bar{g}\langle 22\rangle, \bar{g}\langle 33\rangle] \tag{4.5}$$

with

$$\bar{g}\langle 22\rangle = \bar{g}\langle 33\rangle \tag{4.6}$$

and with $\bar{g}\langle AA\rangle$ being functions of R only. The laminae are the spheres with constant R.

For these three classes of laminated bodies, if the material is incompressible, then we have the following families of statical universal solutions:

(i) *Laminated plate*

Family 0. Homogeneous plane deformations.

$$x = AX, \qquad y = BY+CZ, \qquad z = DY+EZ, \tag{4.7}$$

where (x, y, z) is a rectangular Cartesian coordinate system in the deformed configuration, and where $A, \ldots, E$ are constants satisfying the condition

$$A(BE-CD) = 1 \tag{4.8}$$

for incompressibility.

Family 1. Bending, stretching and shearing of a plate.

$$r = \sqrt{(2AX)}, \qquad \theta = BY+CZ, \qquad z = DY+EZ, \tag{4.9}$$

where (r, θ, z) is a cylindrical coordinate system in the deformed configuration, and where $A, \ldots, E$ satisfy (4.8).

(ii) *Laminated cylindrical shell*

Family 2. Straightening, stretching and shearing a sector of a cylindrical shell.

$$x = \tfrac{1}{2}AR^2, \qquad y = B\Theta+CZ, \qquad z = D\Theta+EZ, \tag{4.10}$$

where $A, \ldots, E$ again satisfy the condition (4.8).

Family 3. Inflation, bending, torsion, extension and shearing of an annular wedge.

$$r = \sqrt{(AR^2+B)}, \qquad \theta = C\Theta+DZ, \qquad z = E\Theta+FZ, \tag{4.11}$$

where the constants $A, \ldots, F$ satisfy the condition

$$A(CF-DE) = 1. \tag{4.12}$$

(iii) *Laminated spherical shell*

Family 4. Inflation or eversion of a sector of a spherical shell.

$$r = (\pm R^3 + A)^{\frac{1}{3}}, \qquad \theta = \pm\Theta, \qquad \varphi = \Phi \tag{4.13}$$

where the constant A is arbitrary.

The meaning of these universal solutions has been discussed thoroughly in Refs. [4] and [5, Sections 57 and 58]. Thus we leave out the detail here.

In Ref. [6], Truesdell has developed a technique for constructing some dynamical universal solutions from any family of statical universal solutions. Specifically, he replaces first the constants A, B, . . . , in (4.7), (4.9), (4.10), (4.11) and (4.13) by functions of t, so that the instantaneous configurations of the motions thus defined are all possible configurations of statical equilibrium. (For this reason, he calls the motions *quasi-equilibrated.*) Next, he requires that the acceleration field $\mathbf{a}$ of the motion be conservative with a single-valued potential ζ, viz,

$$\mathbf{a} = \operatorname{grad} \zeta. \tag{4.14}$$

If this condition is satisfied, then the motion is dynamically possible under the pressure field

$$p = \bar{p} + \rho\zeta, \tag{4.15}$$

where $\bar{p}$ is the pressure field required to maintain the instantaneous configuration in statical equilibrium.

A detailed analysis of the families of dynamically possible quasi-equilibrated motion can be found in Ref. [4]. That analysis remains applicable to the more general class of laminated bodies introduced in this section.

Acknowledgement—This research here reported was carried out under a grant of the U.S. National Science Foundation to Rice University.

REFERENCES

[1] W. NOLL, Materially uniform simple bodies with inhomogeneities, *Arch. ration. Mech. Analysis* **27**, 1–32 (1967).
[2] C.-C. WANG, On the geometric structures of simple bodies, a mathematical foundation for the theory of continuous distributions of dislocations, *Arch. ration. Mech. Analysis* **27**, 33–94 (1967).
[3] C.-C. WANG, Generalized simple bodies, *Arch. ration. Mech. Analysis* **32**, 1–28 (1969).
[4] C.-C. WANG, Universal solutions for incompressible laminated bodies, *Arch. ration. Mech. Analysis* **29**, 161–192 (1968).
[5] C. TRUESDELL and W. NOLL, The non-linear field theories of mechanics, in *Flugge's Handbuch der Physik*, Bd III/3. Springer (1965).
[6] C. TRUESDELL, Solutio generalis et accurata problematum quamplurimorum de motu corporum elasticorum incomprimibilium in deformationibus valde magnis, *Arch. ration. Mech. Analysis* **11**, 106–113 (1962).

(*Received* 7 *April* 1969)

Абстракт—Предлагается общая теория упругих тел, изготовленных из материалов, обладающих заданным типом симметрии, как жидкости, изотропные твердые тела, и др. Специфичные результаты заключают вывод уравнений поля дпя упругих тел с однородной симметрией и некоторые статичные и динамические универсальные решения для несжимаемых изотропных твердых тел.